32 Kai Bluzin hochladen

Übungsstunde lineare Algebra:

Heutige Themen:

- Givensrotation
- QR-Zerl. mit "
- Housholdertransformation
- QR-Zerl. mit "
- LR-Zerl.

$$\begin{bmatrix} a & a & b & b \\ a & a & b & b \\ c & c & d & d \\ c & c & d & d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \underline{A \cdot A + B \cdot C} & \\ & \ddots \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ & \ddots \end{bmatrix}$$

Givensrotation:

$$\underline{\underline{G}}(i,k,\theta) = \begin{array}{c} \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \cos\theta & & \sin\theta & & \\ & & & \ddots & & & \\ & & -\sin\theta & & \cos\theta & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{bmatrix} \begin{matrix} \\ \\ \cdots i \\ \\ \cdots k \\ \\ \\ \end{matrix} \\ \begin{matrix} \qquad\qquad \vdots \qquad\qquad \vdots \\ \qquad\qquad i \qquad\qquad k \end{matrix} \end{array}$$

Gradinarus Notation:

$$D(\varphi) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \leftarrow$$

$$G(\varphi) = D(-\varphi) \leftarrow$$

Bsp:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Drehung im US   Drehung im GUS

QR-Zerlegung mit Givensrotation:

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \Rightarrow \underline{\underline{Q}}\,\underline{\underline{R}}\,\underline{x} = \underline{b}$$

$$\Rightarrow \underline{\underline{R}}\,\underline{x} = \underline{\underline{Q}}^T \underline{b}$$

$$\underline{\underline{L}}\,\underline{\underline{R}}\,\underline{x} = \underline{\underline{P}}\,\underline{b}$$

$$\underline{\underline{A}} = \begin{bmatrix} * & * & * & * \\ \boxed{*} & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \quad = 0$$

$$\underline{\underline{G}}_1 \cdot \underline{\underline{A}} = \begin{bmatrix} * & * & \\ 0 & * & \ddots \\ * & * & \\ * & * & \ddots \end{bmatrix} = \underline{\underline{A}}_2 \,, \quad \underline{\underline{G}}_2 \cdot \underline{\underline{A}}_2 = \begin{bmatrix} * & * & \\ 0 & * & \ddots \\ 0 & * & \\ * & * & \ddots \end{bmatrix} = \underline{\underline{A}}_3 \,, \ldots$$

Beispiel 3.2: V2

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 3 \\ \boxed{-1} & 4 \end{bmatrix}$$

$$\underline{\underline{G}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \underline{\underline{G}} \cdot \underline{\underline{A}} = \begin{bmatrix} * & * \\ \boxed{0} & * \end{bmatrix} = \begin{bmatrix} \cos\theta - \sin\theta & 3\cos\theta + 4\sin\theta \\ \boxed{-\sin\theta - \cos\theta} & -3\sin\theta + 4\cos\theta \end{bmatrix} = \underline{\underline{R}}$$

$$\Rightarrow \theta_1 = 135^\circ, \quad \theta_2 = \underline{315^\circ}, \quad \sin\theta_2 = -\frac{\sqrt{2}}{2}, \quad \cos\theta_2 = \frac{\sqrt{2}}{2}$$

$$\Rightarrow G = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad R = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -1 \\ 0 & 7 \end{bmatrix}, \quad G^{-1} = G^T = Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Allgemein:

$$G_n \cdot G_{n-1} \cdot \ldots \cdot G_1 A = R \iff A = QR = G_1^T G_2^T \ldots G_n^T R$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ \boxed{4} & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad G_1 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G_1 \cdot A = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 9 & 4 \\ \boxed{2} & 1 & 3 \end{bmatrix}, \quad G_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$G_2 \cdot A = \begin{bmatrix} 9 & 2 & 1 \\ 0 & 4 & 7 \\ 0 & \boxed{2} & 3 \end{bmatrix} \quad \cdot - - - - -$$

$$G_3 \cdot G_2 \cdot G_1 A = R \rightarrow A = \underbrace{(G_1^T G_2^T G_3^T)}_{Q} R$$

$$Ax = b \iff QRx = b \Rightarrow \underbrace{Rx = \underbrace{Q^T b}}$$

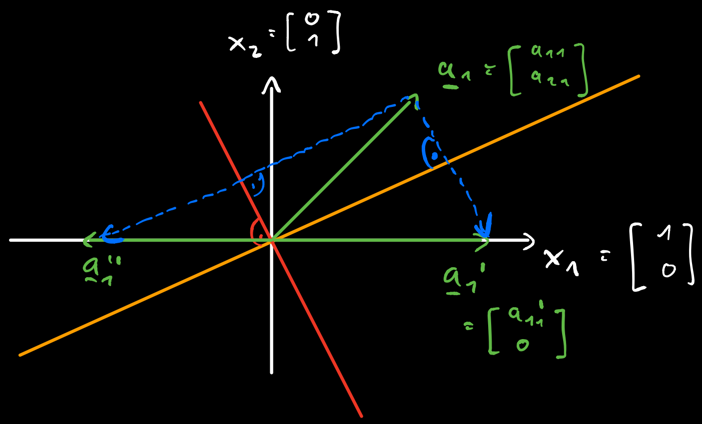
Ergänzung zur LR-Zerlegung -> Zeilenvertauschung:

$$\begin{vmatrix} 1 & \emptyset \\ \emptyset & 1 \end{vmatrix} \begin{vmatrix} 1 & \emptyset \\ \emptyset & 1 \end{vmatrix} A \Rightarrow \begin{vmatrix} 1 & \emptyset \\ \emptyset & 1 \end{vmatrix} \begin{matrix} 1 & & \emptyset \\ 4 & 1 & \\ 5 & 0 & 1 \end{matrix} \begin{vmatrix} 4 & 5 & 6 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} \Rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix} \begin{matrix} 4 & 5 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{matrix}$$

Householdertransformation:

$$H = I - \frac{2}{v^T v} v v^T, \quad v \perp \text{Ebene}, \quad H = I - 2uu^T, \quad u = \frac{v}{\|v\|_2}$$

↳ Normalenvektor



$$\underline{\underline{A}} = \begin{bmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \\ | & | & & | \end{bmatrix}$$

Bsp.: 3.3:

$$\underline{\underline{A}} = \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 2 \\ 1 \end{matrix}} & \begin{matrix} * \\ * \\ * \end{matrix} & \begin{matrix} * \\ * \\ * \end{matrix} \end{bmatrix}, \quad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \circ \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$\underline{a}_{i1} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \| \underline{a}_{i1} \| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\underline{v} = \underline{a}_{i1} \oplus \| \underline{a}_{i1} \| \cdot \underline{e}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \quad \| \underline{v} \| = \sqrt{25 + 4 + 1} = \sqrt{30}$$

$$\underline{u} = \frac{\underline{v}}{\| \underline{v} \|} = \frac{1}{\sqrt{30}} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

$$\underline{\underline{H}} = \underline{\underline{I}} - 2 \cdot \underline{u}\,\underline{u}^T = \begin{bmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} 25 & 10 & 5 \\ 10 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} = \underline{\underline{\frac{1}{15} \begin{bmatrix} -10 & -10 & -5 \\ -10 & 11 & -2 \\ -5 & -2 & 14 \end{bmatrix}}}$$

$$\underline{\underline{H}}\, \underline{a}_{i1} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{H}}_1 \cdot \underline{\underline{A}} = \begin{bmatrix} -3 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix}, \quad \underline{\underline{H}}_2 \cdot (\underline{\underline{H}}_1 \cdot \underline{\underline{A}}) = \begin{bmatrix} -3 & 0 & y \\ 0 & b & y \\ 0 & 0 & y \end{bmatrix}$$

$$\underline{\underline{H}}_3 \cdot \underline{\underline{H}}_2 \cdot \underline{\underline{H}}_1 \cdot \underline{\underline{A}} = \underline{\underline{R}}$$

$$\underline{\underline{A}} = (\underline{\underline{H}}_1^T \underline{\underline{H}}_2^T \underline{\underline{H}}_3^T) \underline{\underline{R}}$$

LR- Zerlegung

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$\underline{\underline{L}} \mid \underline{\underline{R}}$

$$\left.\begin{matrix} 1 & & \emptyset \\ & 1 & \\ \emptyset & & 1 \end{matrix}\right| \begin{matrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 4 \end{matrix}$$

$$\left.\begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{matrix}\right| \begin{matrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & -2 & -2 \end{matrix}$$

$$\underbrace{\left.\begin{matrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{matrix}\right|}_{\underline{\underline{L}}} \underbrace{\begin{matrix} 1 & 1 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{matrix}}_{\underline{\underline{R}}}$$

$$\underline{\underline{L}}\,\underline{\underline{R}} = \underline{\underline{A}}$$

$G \cdot A = R$

$A = G^T R$

$H \cdot A = R$

$A = H^T R$

$$\rightarrow \quad \underline{\underline{B}}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad \underline{\underline{B}}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad \underline{\underline{B}}_2 \underline{\underline{B}}_1 = \underline{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = [$$